AIAA JOURNAL
Vol. 36, No. 7, July 1998

New Structural Dynamic Condensation Method for Finite Element Models

Zu-Qing Qu* and Zhi-Fang Fu†
Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China

A new iterative method is proposed for the reduction of finite element models arising in test–analysis correlation, model updating, finite element modeling, vibration control, structural dynamic optimization, and so on. Based on the modified eigenvalue equation and the eigenvalue shifting technique, two constraint equations for the dynamic condensation matrix, which relates the deformations associated with the master and slave degrees of freedom, are derived. Two iterative schemes are presented for solving the constraint equations. In the second constraint equation, because the dynamic condensation matrix has nothing to do with the eigenpairs of the reduced model, it is unnecessary to calculate them in every iteration. This makes the iterative scheme more computationally efficient than the usual scheme, especially when the number of the master degrees of freedom is large. The accuracy of eigenvalues and eigenvectors of the reduced model is examined in every iterative step. The comparison of the present method with some typical iterative schemes proposed in the past shows that the new one has the highest accuracy. Numerical examples are also presented to show the efficiency of the proposed method.

Nomenclature

D	= $(n \times n)$ dynamic stiffness matrix of a full model, defined in Eq. (4)
D_R	= $(m \times m)$ dynamic stiffness matrix of a reduced model, defined in Eq. (20)
$\bar{D}$	= $(n \times n)$ modified dynamic stiffness matrix of a full model, defined in Eq. (7)
I	= $(n \times n)$ identity matrix
K	= $(n \times n)$ stiffness matrix of a full model
K_R	= $(m \times m)$ stiffness matrix of a reduced model
M	= $(n \times n)$ mass matrix of a full model
M_R	= $(n \times n)$ mass matrix of a reduced model
q	= eigenvalue shifting value
R	= $(s \times m)$ dynamic condensation matrix
ε	= error tolerance
Λ	= $(n \times n)$ eigenvalue matrix of a full model
$\bar{\Lambda}$	= $(n \times n)$ modified eigenvalue matrix of a full model, defined in Eq. (4)
Φ	= $(n \times n)$ eigenvector matrix of a full model
ω	= exact frequency

Subscript

j	= jth frequency or mode

Superscripts

$i, i+1, i+k$	= ith, $i+1$th, and $i+k$th approximation, respectively
T	= transpose
0	= initial approximation

Introduction

DYNAMIC condensation, an efficient method for model reduction, was first applied to large finite element models to provide faster computation of the natural frequencies and mode shapes of a structure. In recent years, it has also been used in test–analysis correlation, model updating, structural damage detection, vibration control, and structural dynamic optimization. Using this method, the stiffness and mass matrices of a full-size finite element model can be condensed to the size of an experimental or reduced model, and the measured mode shapes can also be expanded to the full size of the finite element model.

Probably the first and most popular dynamic condensation method was introduced by Guyan[1] and Irons[2] in 1965. In this method, the inertia terms associated with the slave degrees of freedom (DOFs) are neglected. Hence, the accuracy fully depends on how many and which DOFs are included in the master DOF set.

Dynamic condensation can be broadly classified into three categories: one-step methods,[1–4] two-step methods,[5] and multistep or iterative methods.[6–8] In the iterative approaches, the Guyan condensation matrix is usually defined as an initial approximation. Because the dynamic condensation matrix is updated repeatedly until a desired convergence value is achieved, the selection of master and slave DOFs does not have much effect on the accuracy.

The convergent iterative approach was first proposed by Suarez and Singh[6] in 1992. It is valid for standard eigenproblems, and the general eigenproblems should be transformed into the standard form in advance. The method was modified by Qu and Fu[7] so that it is directly valid for general eigenproblems without Cholesky decomposition. In 1995, Friswell et al.[8] extended the improved reduced system (IRS) method[5] by obtaining the equivalent transformation.

A new iterative method for dynamic condensation is proposed. Two constraint equations for the dynamic condensation matrix are derived directly from the modified eigenvalue equation. Because the constraint equations are nonlinear, two iterative schemes with different convergent criteria are presented at the same time. The convergence of the natural frequencies and modes of the reduced model is examined in every iteration. Two numerical examples, one a cantilevered beam supported by springs and the other a six-story frame, are applied to demonstrate the efficiency of the proposed method.

Constraint Equations for Dynamic Condensation Matrix *R*

In finite element dynamic analysis, the general eigenproblem can be defined as

$$K\Phi = M\Phi\Lambda \tag{1}$$

where M is assumed to be positive definite and K positive semidefinite. The eigenvectors in matrix Φ have been mass normalized, i.e.,

$$\Phi^T M\Phi = I \tag{2}$$

The eigenvalues in matrix Λ are arranged in ascending order. Subtracting $qM\Phi$ from both sides of Eq. (1), one obtains

$$D\Phi = M\Phi\bar{\Lambda} \tag{3}$$

Received Sept. 12, 1997; revision received March 5, 1998; accepted for publication March 9, 1998.

*Ph.D. Student, Vibration Engineering, National Key Laboratory for Vibration, Shock and Noise.

†Vibration Engineering Professor, National Key Laboratory for Vibration, Shock and Noise.

where

$$D = K - qM, \qquad \bar{\Lambda} = \Lambda - qI \tag{4}$$

The eigenvalue shifting value q is usually positive and not equal to any eigenvalues in the eigenvalue range[7] of interest.

Premultiplying both sides of Eq. (3) by the inversion of mass matrix M, one obtains

$$M^{-1}D\Phi = \Phi\bar{\Lambda} \tag{5}$$

Premultiplying both sides of Eq. (5) by matrix D and using Eq. (3) leads to

$$\bar{D}\Phi = M\Phi\bar{\Lambda}^2 \tag{6}$$

where

$$\bar{D} = DM^{-1}D \tag{7}$$

Equation (6) is the modified eigenvalue equation defined in this paper.

Assume that the total DOFs of the full finite element model are divided into master DOFs (m), which will be retained in the eigenproblem equation, and slave DOFs (s), which will be condensed. According to this division and considering the lower mth eigenpairs, Eq. (6) can be partitioned as

$$\begin{bmatrix} \bar{D}_{mm} & \bar{D}_{ms} \\ \bar{D}_{sm} & \bar{D}_{ss} \end{bmatrix} \begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} = \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} \Phi_{mm} \\ \Phi_{sm} \end{bmatrix} \bar{\Lambda}^2_{mm} \tag{8}$$

The subscripts m and s are the parameters related to the master and slave DOFs, respectively. The second equation of Eq. (8) can be expressed as

$$\bar{D}_{sm}\Phi_{mm} + \bar{D}_{ss}\Phi_{sm} = M_{sm}\Phi_{mm}\bar{\Lambda}^2_{mm} + M_{ss}\Phi_{sm}\bar{\Lambda}^2_{mm} \tag{9}$$

Define the dynamic condensation matrix R, which relates the deformations associated with the master and slave DOFs, as[6,7]

$$\Phi_{sm} = R\Phi_{mm} \tag{10}$$

Substituting Eq. (10) into both sides of Eq. (9), one obtains

$$R\Phi_{mm} = \bar{D}^{-1}_{ss}\left[(M_{sm} + M_{ss}R)\Phi_{mm}\bar{\Lambda}^2_{mm} - \bar{D}_{sm}\Phi_{mm}\right] \tag{11}$$

Postmultiplying both sides of Eq. (11) by the inversion of matrix Φ_{mm}, we get

$$R = \bar{D}^{-1}_{ss}\left[(M_{sm} + M_{ss}R)\Phi_{mm}\bar{\Lambda}^2_{mm}\Phi^{-1}_{mm} - \bar{D}_{sm}\right] \tag{12}$$

Based on the same partition form of Eq. (8), Eq. (2) can be rewritten as

$$\begin{bmatrix} \Phi^T_{mm} & \Phi^T_{sm} \\ \Phi^T_{ms} & \Phi^T_{ss} \end{bmatrix} \begin{bmatrix} M_{mm} & M_{ms} \\ M_{sm} & M_{ss} \end{bmatrix} \begin{bmatrix} \Phi_{mm} & \Phi_{ms} \\ \Phi_{sm} & \Phi_{ss} \end{bmatrix} = \begin{bmatrix} I_{mm} & 0 \\ 0 & I_{ss} \end{bmatrix} \tag{13}$$

From the first equation of Eq. (13), one has

$$\Phi^T_{mm}M_{mm}\Phi_{mm} + \Phi^T_{sm}M_{sm}\Phi_{mm} + \Phi^T_{mm}M_{ms}\Phi_{sm} + \Phi^T_{sm}M_{ss}\Phi_{sm} = I_{mm} \tag{14}$$

Substituting Eq. (10) into Eq. (14) yields

$$\Phi^T_{mm}M_R\Phi_{mm} = I_{mm} \tag{15}$$

where

$$M_R = M_{mm} + R^T M_{sm} + M_{ms}R + R^T M_{ss}R \tag{16}$$

and M_R is the reduced mass matrix. Equation (15) is the orthogonal condition of the reduced model. Based on the same derivative procedure, the reduced stiffness matrix K_R and its orthogonal condition are obtained as

$$K_R = K_{mm} + R^T K_{sm} + K_{ms}R + R^T K_{ss}R \tag{17}$$

$$\Phi^T_{mm}K_R\Phi_{mm} = \Lambda_{mm} \tag{18}$$

where K_{mm}, K_{ms}, K_{sm}, and K_{ss} are the submatrices of the stiffness matrix K.

Considering the eigenvalue shifting value q, Eq. (18) can be rewritten as

$$\Phi^T_{mm}D_R\Phi_{mm} = \bar{\Lambda}_{mm} \tag{19}$$

where

$$D_R = K_R - qM_R, \qquad \bar{\Lambda}_{mm} = \Lambda_{mm} - qI_{mm} \tag{20}$$

Postmultiplying both sides of Eq. (19) by the inversion of matrix Φ_{mm}, one has

$$\Phi^T_{mm}D_R = \bar{\Lambda}_{mm}\Phi^{-1}_{mm} \tag{21}$$

Substituting Eq. (21) into Eq. (12) yields

$$R = \bar{D}^{-1}_{ss}\left[(M_{sm} + M_{ss}R)\Phi_{mm}\bar{\Lambda}_{mm}\Phi^T_{mm}D_R - \bar{D}_{sm}\right] \tag{22}$$

Equation (22) is a constraint equation of dynamic condensation matrix R. Obviously, when an iterative scheme is adopted, the physical (stiffness and mass) matrices and the eigenproblem of the reduced model should be calculated in every iteration. Because the eigenproblem analysis is usually time consuming, the solution time used in the iteration rises rapidly as the dimension of the reduced model increases.[9] To reduce the computational work, an alternative constraint equation is defined in the sequel.

Considering Eqs. (15) and (18), the eigenproblem of the reduced model is

$$K_R\Phi_{mm} = M_R\Phi_{mm}\Lambda_{mm} \tag{23}$$

Based on the same derivative procedure of Eq. (6), one obtains

$$D_R M^{-1}_R D_R\Phi_{mm} = M_R\Phi_{mm}\bar{\Lambda}^2_{mm} \tag{24}$$

In both sides of Eq. (24), premultiplying by the inversion of matrix M_R and then postmultiplying by the inversion of matrix Φ_{mm}, one has

$$\left(M^{-1}_R D_R\right)^2 = \Phi_{mm}\bar{\Lambda}^2_{mm}\Phi^{-1}_{mm} \tag{25}$$

Substituting Eq. (25) into the right-hand side of Eq. (12), another constraint equation for dynamic condensation matrix R is defined as

$$R = \bar{D}^{-1}_{ss}\left[(M_{sm} + M_{ss}R)\left(M^{-1}_R D_R\right)^2 - \bar{D}_{sm}\right] \tag{26}$$

There are no eigenvalues and eigenvectors of the reduced model in the constraint equation (26); hence, it is unnecessary to calculate them in every iteration when an iterative scheme is adopted.

Solution of the Constraint Equations

By assuming $\bar{\Lambda}_{mm} = 0$ in Eq. (12), an initial approximate value of the condensation matrix R can be obtained:

$$R^{(0)} = -\bar{D}^{-1}_{ss}\bar{D}_{sm} \tag{27}$$

Based on the initial approximation, two iterative schemes are given in the following.

Scheme 1

1) Choose the eigenvalue shifting q and construct the submatrices of matrices D, K, and M.

2) Calculate the initial approximation of matrix R, K_R, M_R, Φ_{mm}, $\bar{\Lambda}_{mm}$, and D_R using Eqs. (27), (17), (16), (23), and (20), respectively.

3) For $i = 0, 1, 2, \ldots$, begin the iteration as follows. a) Calculate the $i + 1$th approximation of matrix R

$$R^{(i+1)} = \bar{D}^{-1}_{ss}\left[\left(M_{sm} + M_{ss}R^{(i)}\right)\Phi^{(i)}_{mm}\bar{\Lambda}^{(i)}_{mm}\left(\Phi^{(i)}_{mm}\right)^T D^{(i)}_R - \bar{D}_{sm}\right] \tag{28}$$

b) Calculate the physical matrices $K^{(i+1)}_R$ and $M^{(i+1)}_R$ and eigenpair matrices $\Lambda^{(i+1)}_{mm}$ and $\Phi^{(i+1)}_{mm}$ using Eqs. (17), (16), and (23), respectively. c) Check the convergence using the criterion

$$\frac{\left|\omega^{(i+1)}_j - \omega^{(i)}_j\right|}{\left|\omega^{(i+1)}_j\right|} \le \varepsilon \qquad (j = 1, 2, \ldots, p \le m) \tag{29}$$

where $\omega_j^{(i)}$ and $\omega_j^{(i+1)}$ are the ith and $i+1$th approximation of the jth frequency, respectively. Their square values are the eigenvalue $\lambda_j^{(i)}$ and $\lambda_j^{(i+1)}$, which are the ith and $i+1$th diagonal elements of matrix $\Lambda_{mm}^{(i)}$ and $\Lambda_{mm}^{(i+1)}$, respectively. If the given p frequencies converge, exit the loop.

4) Output the results $K_R = K_R^{(i+1)}$ and $M_R = M_R^{(i+1)}$ and stop.

Scheme 2

1) Choose the eigenvalue shifting q and construct the submatrices of matrices D, K, and M.

2) Calculate the initial approximation of matrix R, K_R, M_R, Φ_{mm}, Λ_{mm}, and D_R.

3) For $i = 0, k, 2k, \dots (k > 1)$, begin the iteration as follows. a) Calculate the following equation for k times and obtain the matrix $R^{(i+k)}$

$$R^{(i+1)} = \bar{D}_{ss}^{-1}\left\{\left(M_{sm} + M_{ss}R^{(i)}\right)\left[\left(M_R^{(i)}\right)^{-1}D_R^{(i)}\right]^2 - \bar{D}_{sm}\right\} \tag{30}$$

b) Calculate the matrices $K_R^{(i+k)}$, $M_R^{(i+k)}$, $\Lambda_{mm}^{(i+k)}$, and $\Phi_{mm}^{(i+k)}$. c) Check the convergence using the criterion

$$\frac{\left|\omega_j^{(i+k)} - \omega_j^{(i)}\right|}{\left|\omega_j^{(i+k)}\right|} \le \varepsilon \qquad (j = 1, 2, \dots, p \le m) \tag{31}$$

If the given p frequencies converge, exit the loop.

4) Output the results and stop.

In iterative scheme 1, the eigenvalues and eigenvectors of the reduced model should be calculated one time in one iteration, whereas it is one time for k iterations in scheme 2. Because the eigenvalue analysis is much more computationally expansive than the inverting procedure, scheme 2 is more computationally efficient than scheme 1, especially when the number of the master DOFs is large.

Numerical Examples

To demonstrate the efficiency of the proposed dynamic condensation procedure, the first example, shown in Fig. 1, is a uniform cantilevered beam, which is supported by springs. The stiffness of all of the springs is 5.0E8 N/m. The spaces between all of the neighboring springs are 1.0 m. The beam has a total of 20 DOFs, with 2 DOFs (translational and rotational displacements) at each node. The other properties of the beam are the following: modulus of elasticity 2.0E11 N/m^2, mass density 7800 kg/m^3, area moment of inertia 0.00054 m^4, cross-sectional area 0.2 m^2, and total length 10 m. The lowest four natural frequencies calculated with the full finite element model are 567.31, 579.34, 620.30, and 705.67 rad/s. They are considered as exact values for comparison purposes.

The percent errors of the four natural frequencies are listed in Table 1. The percent error (PE) is defined as

$$\mathrm{PE}\left(\omega^{(i)}\right) = \frac{\omega^{(i)} - \omega}{\omega} \times 100\% \tag{32}$$

When condensed, the translational displacement at nodes 2–5 are selected as master DOFs.

Table 1 PEs of the frequencies calculated with the proposed method (q = 0)

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4
0	2.0903	12.559	40.309	104.47
1	1.0694	4.8467	12.152	29.666
2	0.87253	2.3433	3.6242	9.2033
3	0.77616	1.2450	1.1990	3.4384
4	0.70815	0.87084	0.60664	1.7083
5	0.65157	0.68344	0.38613	1.0499

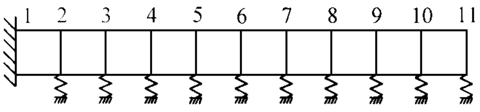

Fig. 1 Cantilevered beam supported by springs.

The results in Table 1 clearly indicate that all natural frequencies will converge to the lower frequencies of the full model monotonically after sufficient iterations. The error of the fourth frequency for the original approximation, for example, is 104.47%. After five iterations, it reduces to 1.0499%, which is about $\frac{1}{100}$th of the original. All of the PEs of frequencies are positive, which indicates that the frequencies of the reduced model are higher than those of the full model. That is, the eigenvalues converge from above.

To examine the convergent characteristics of the eigenvectors, a correlated coefficient for modal vector (CCFMV) value is defined here:

$$\mathrm{CCFMV}\left(\phi_j, \phi_j^{(i)}\right) = \frac{\left|\phi_j^T \cdot \phi_j^{(i)}\right|}{\left\{\left(\phi_j^T \cdot \phi_j\right)\left[\left(\phi_j^{(i)}\right)^T \cdot \phi_j^{(i)}\right]\right\}^{\frac{1}{2}}} \tag{33}$$

where ϕ_j and $\phi_j^{(i)}$ are the exact and ith approximation of the jth ($j = 1, 2, \dots, m$) eigenvector, respectively. A CCFMV value close to 1 suggests that the two modes or vectors are well correlated, and a value close to 0 indicates uncorrelated modes. Obviously, the changes of the elements of the eigenvector $\phi_j^{(i)}$ have no effect on the CCFMV value provided that the shape of the eigenvector does not change, although they do for the vector norms e_j defined in Ref. 6. This makes the CCFMV value more convenient for examination of the accuracy of the eigenvectors than the vector norms. The CCFMV values for the four modes are listed in Table 2.

Obviously, the CCFMV values increase with the iteration steps. For the original approximation, the smallest CCFMV value is 0.15260, which is much closer to 0. After five iterations, it increases to 0.98781, which indicates that the relative modes are well correlated. It is not true that the lower modes always converge faster than the higher modes. The third mode, for example, converges faster than the first and second modes.

As stated in Ref. 7, the eigenvalue shifting technique can improve the accuracy of the frequencies and modes of the reduced model for a given iteration. This is proved by the results in Tables 1–4. The largest PE in Table 3, for example, is 0.68005% after two iterations, whereas it is 1.0499% after five iterations in Table 1. Similarly, the CCFMV values in Table 4 are much higher than those in Table 2.

Table 2 CCFMV values of the modes calculated with the proposed method (q = 0)

Iteration	Mode 1	Mode 2	Mode 3	Mode 4
0	0.97616	0.39944	0.21771	0.15260
1	0.98904	0.89217	0.70967	0.55581
2	0.99111	0.96446	0.95167	0.90294
3	0.99199	0.97930	0.99083	0.96644
4	0.99262	0.98571	0.99502	0.98177
5	0.99317	0.98951	0.99648	0.98781

Table 3 PEs of the frequencies calculated with the proposed method (q = 280,000)

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4
0	0.36533	1.6581	9.7166	54.199
1	0.01777	0.02072	0.10534	3.7020
2	0.01615	0.02041	0.01446	0.68005
3	0.01568	0.01996	0.00465	0.22739
4	0.01536	0.01953	0.00185	0.09763
5	0.01506	0.01910	0.00082	0.04700

Table 4 CCFMV values of the modes calculated with the proposed method (q = 280,000)

Iteration	Mode 1	Mode 2	Mode 3	Mode 4
0	0.99878	0.98213	0.76875	0.32113
1	0.99984	0.99990	0.99980	0.97849
2	0.99981	0.99989	0.99992	0.99592
3	0.99980	0.99988	0.99996	0.99849
4	0.99980	0.99988	0.99998	0.99932
5	0.99980	0.99988	0.99999	0.99967

Table 5 PEs of the frequencies calculated with the method of Ref. 6

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4
0	3.7864	24.030	76.872	178.76
1	2.1063	12.658	40.687	105.67
2	1.6401	8.9491	25.568	61.405
3	1.4308	6.9538	15.779	32.875
4	1.3100	5.6358	9.3855	18.489
5	1.2205	4.5131	5.2880	10.938

Table 6 PEs of the frequencies calculated with the methods of Refs. 7 and 8

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4
0	3.7700	24.013	77.187	176.97
1	2.0903	12.559	40.309	104.47
2	1.6269	8.8831	25.396	61.013
3	1.4189	6.9046	15.716	32.765
4	1.2990	5.5949	9.3468	18.393
5	1.2102	4.4806	5.2637	10.874

Table 7 CCFMV values of the modes calculated with the method of Ref. 6

Iteration	Mode 1	Mode 2	Mode 3	Mode 4
0	0.94626	0.43113	0.01566	0.06585
1	0.98113	0.47288	0.29154	0.10227
2	0.98744	0.69052	0.49608	0.19743
3	0.98968	0.80363	0.67795	0.59127
4	0.99091	0.86504	0.82980	0.81067
5	0.99179	0.90483	0.92664	0.90718

Table 8 CCFMV values of the modes calculated with the methods of Refs. 7 and 8

Iteration	Mode 1	Mode 2	Mode 3	Mode 4
0	0.93468	0.02150	0.07883	0.06925
1	0.97616	0.39944	0.21771	0.15260
2	0.98380	0.62756	0.43142	0.09739
3	0.98649	0.75367	0.62114	0.48959
4	0.98796	0.82541	0.78372	0.74033
5	0.98901	0.87350	0.89565	0.85930

The effects of the eigenvalue shifting technique are discussed in detail in Ref. 7.

To show the advantages of the proposed dynamic condensation method, the PEs of frequencies and CCFMV values of modes obtained from the approaches of Refs. 6–8 are also listed in Tables 5–8. Four conclusions can be drawn from these results. 1) The methods of Refs. 7 and 8 provide exactly the same results. That is, the approach that is extended from Suarez and Singh[6] to validate generalized eigenproblems directly is equivalent to the iterative IRS approach. 2) There is little difference between the results obtained from the method of Ref. 6 and that of Refs. 7 and 8. Because the Cholesky decomposition of the full mass matrix and the inversion of the matrix S are required in Ref. 6, the former is more computationally expensive than the latter. 3) The accuracy of the frequencies and modes calculated with the proposed method is much higher than the methods of Refs. 6–8. After five iterations, the biggest PEs of frequencies obtained from the proposed method, the method of Ref. 6, and the methods of Refs. 7 and 8 are 1.0499%, 10.938%, and 10.874%, respectively. The first is less than $\frac{1}{10}$th of the latter two. 4) The PEs and the CCFMV values of the original approximation obtained from the proposed method with $q = 0$ are equal to those of the first approximation from the methods of Refs. 7 and 8 with $q = 0$.

The proposed dynamic condensation method has also been tested on a six-story structure, shown in Fig. 2. The structure is modeled as a two-dimensional frame. It has a total of 32 nodes, with the first 2 nodes grounded. The structure has 90 DOFs. For all of the beams, the modulus of elasticity is 2.1E11 N/m^2, mass density 7830 kg/m^3, area moment of inertia 0.00054 m^4, and cross-sectional area 0.2 m^2. The height and width of all of the stories are 1.0 and 2.0 m, respectively. The first six natural frequencies obtained from the full finite element model are shown in Table 9. These values are considered as the exact values for comparison purpose.

The horizontal displacements at nodes 18, 20, 23, 25, 28, and 30 are selected as master DOFs when condensed. The PEs of the first six frequencies for $q = 0$ and 5500 are listed in Tables 10 and 11, respectively.

The results in Tables 10 and 11 also indicate that the frequencies of the reduced model converge from above with iteration. Although the eigenvalue shifting technique can improve the accuracy of frequencies and modes for given iterations, the effects decrease rapidly

Table 9 Natural frequencies of the six-story frame in Fig. 2

Freq. number	Natural freq., rad/s
1	80.653
2	260.92
3	490.82
4	770.88
5	812.14
6	1052.5

Table 10 PEs of the frequencies calculated with the proposed method ($q = 0$)

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4	Freq. 5	Freq. 6
0	0.00001	0.08964	3.4036	20.412	25.156	12.937
1	0.00000	0.00727	0.23993	2.4801	10.040	6.4629
2	0.00000	0.00287	0.09056	0.94153	3.1190	4.1599
3	0.00000	0.00137	0.04313	0.48343	15.176	3.0172
4	0.00000	0.00072	0.02283	0.27970	0.96030	2.3885
5	0.00000	0.00041	0.01291	0.17269	0.68211	2.0031

Table 11 PEs of the frequencies calculated with the proposed method ($q = 5500$)

Iteration	Freq. 1	Freq. 2	Freq. 3	Freq. 4	Freq. 5	Freq. 6
0	0.00000	0.06622	3.2026	20.144	25.049	12.869
1	0.00000	0.00539	0.22474	2.4192	9.8524	6.4260
2	0.00000	0.00211	0.08435	0.91443	3.0488	4.1353
3	0.00000	0.00100	0.03990	0.46705	1.4876	3.0004
4	0.00000	0.00053	0.02099	0.26875	0.94236	2.3761
5	0.00000	0.00030	0.01180	0.16501	0.66944	1.9933

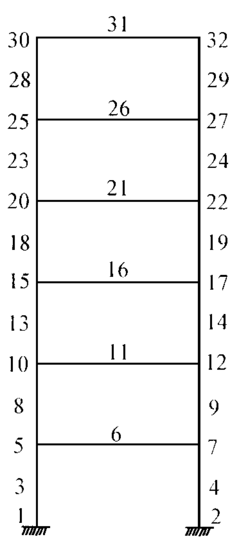

Fig. 2 Schematic of a six-story frame.

with the increasing of frequencies for the structures in which the lower frequencies are very scattered.

Conclusions

A new dynamic condensation method for finite element models is proposed. Two constraint equations for the dynamic condensation matrix are derived directly from the modified eigenvalue equation. One iterative scheme with a convergent criterion for each of the constraint equations is presented at the same time. The accuracy of frequencies and eigenvectors of the reduced model is examined during iteration. The following four conclusions can be drawn from the new method.

1) All natural frequencies and modes of the reduced model will converge to the relative ones of the full model after sufficient iterations. The frequencies converge from above.

2) The frequencies and modes calculated with the present method are more accurate than three typical iterative schemes for dynamic condensation.

3) Because it is unnecessary to calculate the eigenvalues and eigenvectors of the reduced model in every iteration in iterative scheme 2, the scheme is more computationally efficient than scheme 1, especially when the dimension of the reduced model is large.

4) The eigenvalue shifting technique can improve the accuracy of the frequencies and modes of the reduced model for given iterations. It can be used for structures where approximations of the lower frequencies are available. However, when the lower frequencies are very scattered, the technique does not have much effect on the accuracy of higher eigenpairs.

Acknowledgment

Special thanks to Zhao Cai for sustaining encouragement.

References

[1] Guyan, R. L., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, No. 2, 1965, p. 380.

[2] Irons, B., "Structural Eigenvalue Problems—Elimination of Unwanted Variables," *AIAA Journal*, Vol. 3, No. 5, 1965, pp. 961, 962.

[3] Johnson, C. P., Craig, R. R., Jr., Yargicoglu, A., and Rajatabhothi, R., "Quadratic Reduction for the Eigenproblem," *International Journal for Numerical Methods in Engineering*, Vol. 15, No. 6, 1980, pp. 911–923.

[4] Paz, M., "Practical Reduction of Structural Eigenproblems," *Journal of Structural Engineering*, Vol. 109, No. 11, 1982, pp. 2591–2599.

[5] O'Callahan, J. C., "A Procedure for an Improved Reduced System (IRS) Model," *Proceedings of the 7th International Modal Analysis Conference* (Las Vegas, NV), Union College, Schenectady, NY, 1989, pp. 17–21.

[6] Suarez, L. E., and Singh, M. P., "Dynamic Condensation Method for Structural Eigenvalue Analysis," *AIAA Journal*, Vol. 30, No. 4, 1992, pp. 1046–1054.

[7] Qu, Z.-Q., and Fu, Z.-F., "An Iterative Method for Dynamic Condensation of Finite Element Models, Part I: Basic Method," *Journal of Shanghai Jiao Tong University*, Vol. 3, No. 1, pp. 18–24 (English ed.).

[8] Friswell, M. I., Garvey, S. D., and Penny, J. E. T., "Model Reduction Using Dynamic and Iterated IRS Techniques," *Journal of Sound and Vibration*, Vol. 186, No. 2, 1995, pp. 311–323.

[9] Qian, Y., and Dhatt, G., "An Accelerated Subspace Method for Generalized Eigenproblems," *Computers and Structures*, Vol. 54, No. 6, 1995, pp. 1127–1134.

A. Berman
Associate Editor